Invited Lecture

Mathematical Argumentation, a Precursor Concept of Mathematical Proof

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ABSTRACT This lecture offers a reflection on the challenge posed by the current trend of curricula and standards to recommend starting the learning of proof from the very beginning of the compulsory school. This trend pushes on the fore the notion of argumentation, it is here discussed as well as its relations to proof as a convincing and an explaining legitimate means to support the truth of a statement in the mathematics classroom. Eventually, a didactical concept of mathematical argumentation is discussed and elements of its characterization are proposed.

Keywords: Mathematical argumentation; Early learning of proof; Epistemology.

1. Early Learning of Mathematical Proof

While "mathematical proof"[2] disappears from the mathematics teaching challenges of the 21st century compulsory school, learning how to back the truth of a statement in the mathematics classroom is still on the fore with the concept of "proof":

> The notion of proof is at the heart of mathematical activity, whatever the level (this assertion is valid from kindergarten to university). And, beyond mathematical theory, understanding what is a reasoned justification approach based on logic is an important aspect of citizen training. The seeds of this fundamentally mathematical approach are sown in the early grades (Villani and Torossian, 2018, pp. 25–26 — free translation).

Since it is meant to cover all grades, "proof" is used here with its vernacular meaning. The expression of this objective takes different form in curricula, using a variety of expressions: *deductive reasoning*, *proof*, *justification*, *mathematical argumentation,* etc.

Since 2003, the TIMSS[3] assessment frameworks provide a picture of the way proof and proving have evolved since the beginning of the 21st century. They distinguish

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[2] "Mathematical proof" is used to translate the words used by Roman language which etymology is the Latin "demostratio" (e.g. démonstration in French). "Demonstation" was used by Anglophone mathematicians until the beginning of the 20th century.

[3] "Trends in International Mathematics and Science Study" which gives a consensus picture of the common core competencies for 4th and 8th graders. https://timssandpirls.bc.edu/timss2003i/frameworksD.html

content domains (the mathematics subject matter) from *cognitive domains* (the behaviours expected from students). Issues related to validating a mathematical statement are addressed in the sub-domain "reasoning" of the cognitive domain:

> "Reasoning mathematically involves the capacity for logical, systematic thinking. It includes intuitive and inductive reasoning based on patterns and regularities than can be used to arrive at solutions to non-routine problems. […] reasoning involves the ability to observe and make conjectures. It also involves making logical deductions based on specific assumptions and rules, and justifying results."

Reasoning includes several skills among which *Justify*; this keyword was associated to *Prove* in the 2003 assessment framework, it disappeared from the following assessment campaigns. Then, the reference to mathematical proof being abandoned, the keyword which is chosen is "justification" with specific requirements: "reference to mathematical results or properties" (TIMSS 2007, 2008, 2011). Then comes back the key expression "mathematical argument" (TIMSS 2015, 2019) in a short and allusive statement.

Researchers in mathematics education are fully seized of the problems of teaching proof, witnessing its fundamental character for the learning of mathematics. The number of articles and conference communications has impressively increased since the pioneer work of Alan Bell (1976). One of the first collective book "Theorems in School" (Boero, 2007) deserves a special attention. Its idea was born in the context of the 21st PME conference which demonstrated "the renewed interest for proof and proving in mathematics education" and that of "important changes in the orientation for the curricula in different all over the world" (ibid. p. 20).

In 2007, ICMI launched its 19th study on "Proof and proving in mathematics education" (Hanna and de Villiers, 2012/2021). As learning to justify/prove was to be addressed since the early grades, the idea of proof had to be extended. The study introduced the idea of "developmental proof" as "a precursor for disciplinary proof (in its various forms) as used by mathematicians" (ibid. p. 444). The introduction of this idea intended (1) to provide "a long-term link with the discipline of proof shared by mathematicians", (2) to provide "a way of thinking that deepens mathematical understanding and the broader nature of human reasoning", (3) to "gradually developed starting in the early grades" (ibid.)

The introductory discourse of these initiatives reflects a complexity we know since the seminal exploratory study of Harel and Sowder (1998), covering a large spectrum from "external conviction proof schemes" to "analytical proof schemes" (ibid. p. 245). I will not address all this complexity here, instead I will focus on the educational project aiming at developing the early acquisition of the competence of arguing (to convince) and of proving (to establish the truth). *The didactical project is to teach how to respond to the question of truth and to understand the role of proof in mathematics.*

Proof is a difficult concept per se. We discussed it at length. But, not surprisingly as mathematicians, we didn't discuss the concept of truth. Maybe we should have. I will address this issue in the next section. Then, I will consider some terms of the related vocabulary with the objective to propose elements for a characterization of the concept of *mathematical argumentation* as a precursor for the transition to *mathematical proof*.

2. Are We Sure of What "True" Means?

Proof and *truth* are inseparable concepts, yet discussions on what can count as proof proceed as if the meaning of the word *truth* were clear. This may seem an irrelevant issue in mathematics where *true* and *false* are just the elements of set where propositions or predicates take value. But mathematical logic is not the logic of mathematics insofar as the activity of mathematicians is not reduced to carrying out a formalism. "Actually, the criterion of truth in mathematics is the success of its ideas in practice; mathematical knowledge is corrigible and not absolute; thus, it resembles empirical knowledge in many respects", wrote Hilary Putnam (1975, p. 529) in a brief paper entitled *What is mathematical truth?* . This position is rather radical, but it is relevant for our topic: *more than a science, in the K-9 mathematical classroom, mathematics is a practice*.

In school, the words *true* first borrows its meaning from the vernacular culture. If students in higher education maintain a difference between the mathematical meanings of true and its meaning in everyday life, this is not the case for K-9 students. In thinking about this problem, I wondered whether we share the meaning of true and truth? To get a glimpse of an answer, I looked at the case of writing in English something thought in French: is the direct translation of the French *vrai* by the English *true* without consequence?

The etymology[4] of *true* goes back to the word *tree*, which denotes firmness or faithfulness. Its evolution incorporated other meanings among which the mathematical one (i.e. logical necessity). Still, the contemporary use puts sincerity and reliability ahead of veracity. The etymology of *vrai* goes back to the Latin word *veritas* whose paradigm is normative: it refers to the legal truth that a legitimate institution locks and preserves. The evolution has introduced the producer of the statements claimed true, of his or her sincerity, but the normative meaning still dominates.

This issue concerns all the languages and background cultures of our research projects. The epistemological differences silently shape research. Eventually, the investigation which started by noticing possible translation issues ends up inviting us to consider the vernacular epistemology. The tension between vernacular languages and mathematical language should lead to *questioning the culture that proof and truth* carry with them.

Davidson (1996) warned us that *it is folly to try to define truth*. But the word-concept *proof* is inseparable from the word-concept *true*. In agreement with Durand-Guerrier (2008, p. 373), I turn to Alfred Tarski's solution to chose "a definition which

[4] According to the Vocabulaire Européen des Philosophies (Cassin, 2004)

is materially adequate and formally correct.", he defines "truth and falsehood simply by saying that a sentence is true if it is satisfied by all objects, and false otherwise"[5]. But he adds the condition that sentences are elements of "[a language] whose structure has been exactly specified." (Tarski, 1944, p. 341, 347 & 373).

In order to take the consequence of this condition, let us introduce the distinction made by John Langshaw Austin (1950) between *statement* and *sentence*. The utterance of a *statement* requires words and a good command of linguistic rules to produce a *sentence* appropriate to the communication objective which underpins it. This objective includes semantic adequacy and formal correctness. Furthermore, Austin introduces a speaker and an audience, in other words *the intentional character of the speech act uttering truth, and its social dimension*. Hence, aside from coherence and correspondence, the hypothesis of sincerity and steadfastness of the speaker and of the audience must be included.

Although somewhat limited, this discussion sheds light on the difficulty of comprehending the meaning of truth when taking a step beyond mathematical logic while remaining within the mathematical territory. Mathematics as a scientific discipline is universal. Mathematical activity is diverse, it embraces the cultural and historical characteristics of the society in which it develops. This is even more so for its learning and teaching, which are situated mathematical activities framed by institutions and political projects of a society.

Then, I propose four conditions to consider *the truth of a sentence*:

- to be ethically minded (sincerity, reliability)
- to be linguistically appropriate (statement vs sentence)
- to be semantically adequate (correspondence)
- to be formally correct (coherence)

These conditions will not have the same importance within the transition from argumentation at the earliest learning stages to mathematical proof. Nevertheless, we ought to take on such epistemological and didactical perspectives to revisit the classical issue of defining proof in mathematics fitting the needs of mathematics teaching.

3. Reasoning, Explanation, Argumentation and Proof

3.1. Reasoning

The general framework within which problem solving and proving are studied is under the common umbrella of the word "reasoning", which often denotes the mental process of making inferences. I used such a definition for my early work. On reflection, this formulation was awkward because it directed attention to the modelling of mental processes, whereas the problem posed to the teacher is that of *the mathematical interpretation* of observed behaviour and productions. Then, I turned myself to

[5] Tarski's definition grounds the deduction theorem which bridges syntax and semantic, truth and validity.

Raymond Duval's definition which refers to tangible expressions of thought. It makes the analysis of reasoning a work on discourses and texts whose contextualisation by the state of knowledge, the levels of language and the constraints of the situation are considered:

> Reasoning is the organisation of propositions which is directed towards a target statement in order to modify the epistemic value that this target statement has in a given state of knowledge, or in a given social environment, and which, as a consequence, modifies its truth value when certain particular conditions of organisation are met (Duval, 1992, p. 52 — free translation).

By "particular conditions of organisation", Duval refers to both the logical structure and the particular norm of the proof discourse. This definition, on the one hand, satisfies our theoretical needs, on the other hand, it does not introduce contradiction with a psychological approach.

3.2. Explanation

Gila Hanna pioneered the discussion on the distinction between *proof that proves* and *proof that explains*. It refers to *the question of why* a statement is true, which is that of the link between proof and knowledge.

Duval, did not miss these distinctions: "once the question of epistemic value has been resolved, the question of the construction of coherence or belonging of the new production to the system of knowledge arises" (ibid. p. 40). At the end of the problem-solving process, the explanation is thus the explicit system of relations of the stated result with the available knowledge of the problem-solver. The related proof will have an explaining value if this system is congruent with the knowledge of the interlocutors. This approach is reasonable and productive in our domain, but it induced Duval to assert a division between explanation and reasoning (to justify). The former, he wrote (ibid. pp. 37, 39 & 51), gives one or more reasons to make a result understandable, whereas for the latter the role of the reasons put forward is to communicate to the statements "their strength of argument"; that is to say: their role is to convince.

In claiming the existence of such a division, Duval induces one between explanation and proof that Hanna rejects:

> "A proof becomes legitimate and convincing for a mathematician only if it leads to a real mathematical understanding." (Hanna, 1995, p. 42).

To deepen this issue, it is interesting to return to the term "argumentation".

3.3. Argumentation

One always comes to argumentation with a substantial knowledge of what argumentation is, remarks Christian Plantin. In addition to the common-sense conceptions of argumentation, several disciplines contribute to its meaning, among which philosophy, logic, cognitive sciences, linguistic. For the issue addressed here, I will focus on the contribution of linguistic. Within this discipline, there is not a single approach of argumentation, it is therefore advisable to specify this word in order to have an effective characterisation and move forward without creating insurmountable conflicts.

In common use, the term argumentation designates both the action of arguing and its product. The associated process implements linguistic and representational means to make possible interactions (actual or potential) between protagonists who seek to ensure the validity of a statement or, on the contrary, oppose and confront their positions. The outcome takes the form of a discourse that materializes the reasons for agreement or disagreement. In order to distinguish between the process and the product, I will use the verb "to argue" to refer to the former, and the noun "argumentation" for the latter. Drawing on Plantin (1990) synthesis, I suggest the following characterisation:

Argumentation is a discourse

- *Oriented*: it aims at the validity of a statement;
- *Critical*: it analyses, supports and defends;
- *Intentional*: it seeks to modify a judgment.

Arguing is a process

- Which instruments the language;
- Which changes the epistemic value of a judgment;
- Which changes the relationship to knowledge[6].

This distinction is congruent with that made by Duval between rhetorical argumentation and heuristic arguing (ibid. p. 51). The former aims at convincing an interlocutor, whereas the latter emphasizes the role of arguing in guiding problem-solving. This distinction makes it possible to bring the common understanding of argumentation closer to one that is congruent with the requirements of a mathematical activity. Then, an argumentation is accepted or rejected according to two criteria: its relevance (semantic coherence) and its epistemic value (strength of a belief).

Moreover, the concept of epistemic value facilitates shaping the difference between mathematical argumentation and mathematical proof. The reference to the epistemic value induces the idea of its dependence to an author, whereas the value of a mathematical statement depends on the mathematics not on the mathematicians.7 There is a possibility of thematizing this opposition (Hanna, 2017) by taking up the distinction made by the philosophers Frans Delarivière and Bart van Kerkhove between epistemic value, which implies the existence of an agent, and ontic value, which is independent of any agent. For these authors, it was a question of qualifying the intrinsic or relative character of the explanatory value of a proof. Here is what they write:

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[6] Knowledge refers here to the pair {statement, argument}

[7] I don't ignore the pragmatic limit of such a claim since mathematics is the product of a human activity.

> "A mathematical proof can be thought of as an argument by which one convinces oneself or others that something is true, so it may seem difficult to go beyond epistemic discourse about an explanatory proof. However, if the content of any particular piece of evidence is the product of one person's epistemic work, it can be separated as an object independent of a particular mind. Other people can read this evidence and be convinced of it. This brings us to the question of whether showing why a theorem is true is a feature of the proof itself or a feature of communicative acts, texts or representations." (Delarivière et al., 2017, p. 3)

This is to be compared with the criterion for recognising the heuristic or epistemic character of an argument, "[which] is either due to the existence of a theoretical organisation of the field of knowledge and representations in which the argumentation takes place, or to the absence of such a theoretical organisation." "A heuristic argumentation requires the existence of a theoretical organisation of the field of knowledge and representations in which the argumentation takes place" and "that one is able to understand or produce a relation of justification between propositions that is deductive and not only semantic in nature" (Duval, 1992, p. 51 & 52).

Thus, the distinction between rhetorical argumentation and heuristic argumentation comes down to the evaluation of the epistemic value and the ontic value of statements. We can then argue that *an argumentation will be admissible in the sense of mathematics if the epistemic value of its statements is conditioned by their ontic value*. It is this criterion that will allow it to be recognised as a proof in mathematics. The mathematical normalisation of proofs is a technical means of carrying out this evaluation.

3.4. Proof

We have learned that the epistemological journey from argumentation to mathematical proof is long and full of pitfalls. The first issues were on proof and logic, then on the relation between explanation and proof, and between proof and mathematical proof. Argumentation became later a research theme with the idea of a fundamental conflict between argumentation and proof. The former could be seen as an epistemological obstacle to the latter. I support this idea. But I see a solution to the problem it raises, which is to give room to the concept of *mathematical argumentation* in the that of *developmental proof*. The distinction between rhetorical and heuristic argumentation, and between epistemic and ontic value, makes it possible to progress in this direction.

Following Duval, the tension between argumentation and mathematical proof originates in the nature of inferences which *could be* of a semantic in the first case and *must be* of a logical in the second case (Fig. 1, on the next page). It suggests a shift in the learner's position from a pragmatic stance to a theoretical stance (Balacheff, 1990). An adequate characterization of mathematical argumentation should be a tool to facilitate this evolution.

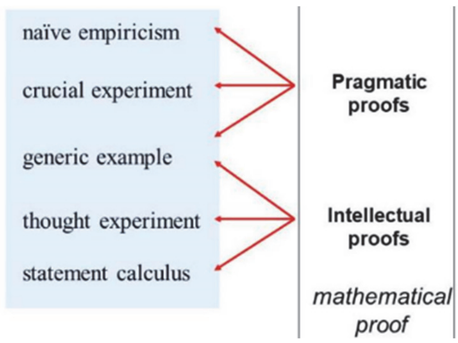

Fig. 1.

My research questioned what could be considered as a proof for students *before* they were formally introduced to the Euclidean norm of mathematical proof. It led me to distinguish between pragmatic and intellectual proofs, and within each category to identify different types of proof. The outcome of this initial research was that the type of proof is determined in the first place by the nature of the students knowing and their available semiotic representation. In the private space, the effort is to construct an argumentation which at is both convincing and meaningful. It is in the context of a social interaction that argumentation may take precedence over explanation. The split between them could cover the large range of the possible proof schemes. Indeed, social interaction cannot be avoided; it is a source of complex phenomena that the teacher has to manage.

Then, what comes first is an "explanation" of the validity of a statement from the student's own perspective, without prejudging *what counts for her or him as an explanation*, whether in terms of content or of form of the text which expresses it. The rationale for this postulate is that the explaining power of a text is directly related to the quality and density of its roots in the learner's knowing. So, the key issue of an approach of the learning of proof is that of the nature of the relation between the students' knowing and their argumentation supporting the validity of a statement.

The passage from explanation to argumentation is imposed by the need to communicate reasons and their organisation. Having others accept that an argumentation establishes the validity of a statement changes its status, it becomes public and gets the status of proof.

The important point is to highlight the existence of a boundary between the private and public spaces. In the private space, explanation works on objects and their relations, it is the basis for the construction of the explanation which backs the validity of the solution of a problem, whether or not this work ensures the submission of epistemic value to ontic value. Crossing this boundary implies the search for a consensus. This social process, by its very nature, cannot guarantee that the protagonists individually recognise the explanatory character of the collectively accepted argumentation — *the proof*. This uncertainty is even greater in the case of mathematical proof because of its normative character which takes precedence over its rhetorical characteristics.

4. Three Short Stories and One Lesson

This section presents three examples intending to illustrate aspects I will later consider in order to characterize mathematical argumentation from a didactical perspective. They deal with the relation between knowing, semiotic resources and controls as tools in a validation process. I start with the case of a famous mathematician, so that we realize that the issue is not only that of beginners but in a way intrinsic to mathematics.

4.1. Short story 1, where rationality and cognitive maturity are not the issues

In his *Cours d'analyse*[8] published in 1821, Augustin Cauchy formulated a first version of a theorem on the convergence of series of continuous functions:

> Let (I) "$u_0,\ u_1,\ u_2, \dots, u_n,\ u_{n+1}, \dots$" be a series, then the theorem states:
>
> "When the various terms of series (I) are functions of the same variable x, continuous with respect to this variable in the neighbourhood of a particular value for which the series converges, the sum s of the series is also a continuous function of x in the neighbourhood of this particular value." (trans. Bradley and Sandifer 2009 p.90)

As we now know, this statement is false. Cauchy recognized its refutation by other mathematicians. He modified it and published a new modified statement in the *Comptes rendus à l'Académie des Sciences*, thirty years after the first edition of the course, in 1853. Why such an outstanding mathematician didn't realize the error he was making once refutations were known, and why was it so difficult to overcome it?

Gilbert Arsac (2013) studied this episode paying attention to avoiding anachronisms which could introduce the rewriting of Cauchy's writings with the formalization of the contemporary mathematics. Such rewriting would have hidden the conceptual difficulties mathematicians met, especially with the notions of *function* and *variable*.

Arsac first points that the variable x is not explicit in the expression (I), although the modern notation $f(x)$ was used in the course. In fact, in this expression, $\mathrm{u_n}$ and x are two variables, x being the independent variable on which depends the functions $\mathrm{u_n}$. Second, he reminds us that the dominant concept image of *limit* is cinematic, reinforced by the role drawing the curve of functions played. Then validity of the theorem was established using a narrative which expressed a qualitatively the reasoning. Here is an extract:

> [let s_n be the partial sum as rank n, r_n the reminder and s the limit, these] three functions of the variable x, the first of which is obviously continuous with respect to x in a neighbourhood of the particular value in question. Given this, let us consider the increments in these three functions when we increase x by an infinitely small quantity α. For all possible values of n, the increment in s_n is an infinitely small quantity. The increment of r_n, as well

[8] http://gallica.bnf.fr/ark:/12148/btv1b8626657

as r_n itself, becomes infinitely small for very large values of n. Consequently, the increment in the function s must be infinitely small." (Bradley and Sandifer, 2010, p. 89–90)

However, Cauchy did not present this text as a mathematical proof as he did for other theorems in his course, but as a *remark*. This remark invites the reader to imagine with the mathematician the monotonous movement of x and the effect it causes on the functions at each step of the reasoning. Things happen because they "*must*" happen.

The 1853 proof introduced the criterion of uniform convergence:

$s_{n'} - s_n = u_n + u_{n+1} + \cdots + u_{n'-1}$ becomes infinitely small for infinitely large value of the numbers n and $n' > n$.

But still, this proof has the style of a narrative dominated. The order of the statements and the appearance of the terms driven by the rhetoric of argumentation is not congruent with the logical order of the formal n/ε proof. As it were, it hides the dependence of n on ε and not on x, as it is evidenced by the modern algebraic expression[9], is. The style of the Cauchy's revised version is still to that of the initial remark, however he now calls it a mathematical proof.

The will to be rigorous is undoubtedly present throughout Augustin Cauchy's work, but it encounters obstacles: the definitions of *variable* and *function*, the absence of *the sign* $<$ and hence the formal manipulation of inequalities, the absence of a notation for *absolute value* (introduced by Weierstrass in 1841) and of the *quantifiers* (introduced at the turn of the 20th century). Eventually, the natural language is infused by a cinematic concept image of *convergence* and the Leibnizian "*lex continuitatis*" (law of continuity).

Gilbert Arsac analysis evidences the tight relation between *representation*, *language* and the *reasoning tools* on the one hand, and on the other hand the limits due to the cinematic conception of *continuity* and *limit*. The difficulty of Cauchy was not due to his underlying rationality and his cognitive maturity.

4.2. Short story 2, where it is question of semantic control

It is common to observe that students' early learning of geometry meets difficulties with the concepts of perimeter and area, and their relations. I studied some of these difficulties met by 7th and 8th graders, using a classical task about the perimeter and the area of a rectangle which is a familiar object for them. They know a lot about it, either as a geometrical object or as a shape for which they can calculate the area and perimeter. The task consists of asking students working in pairs what they think of certain claims attributed to other students. I take the case of a pair, A&C, about two of these claims:

Serge: if you increase the area of a rectangle its perimeter also increases.

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[9] $\forall \varepsilon\ \exists N\ \forall n > N\ \forall n'\ [n' > n \rightarrow \forall x\ |s_n - s_{n'}| < \varepsilon]$

> *Brigitte:* all rectangles that have an area of 36cm² have a perimeter that is not less than 24cm.
>
> What do you think of what each of these students say: do you agree or disagree? Explain why.

A&C judged positively Serge's proposition, but students did not see at once how to explain it: *"It's silly because it's obvious [...] how can we prove it?"* They return to this question after considering Brigitte's claim which induces to use the area and perimeter formula. Without changing their initial judgment, they invoke arithmetic properties:

> "When you increase the perimeter, the numbers you increase them ... there, the numbers that multiply ... that add up [...] well yes, because when you increase the perimeter, the length and width increase. So when you multiply them both, it increases too."

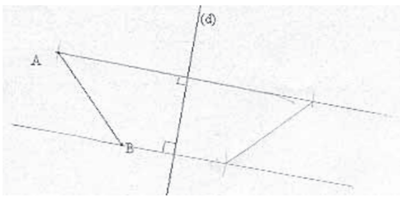
The A&C case illustrates an area-perimeter conception that develops within the framework of symbolic arithmetic in which formulas provide a representation whose manipulation and interpretation is under the control of their referent (i.e., what they model). The principle of a monotonously increasing covariation of area and perimeter is strong enough to impose itself and control the manipulation of the formulas. In both cases, students were not limited by the semiotic tools needed to achieve the proposed task, nor by logical skills. Their search was bounded by their conceptions.

4.3. *Short story 3, where the issue is the restructuration of knowledge*

Although students seem to master some mathematical tools, the way they use them in different situations may reveal inconsistency which could leave the teacher wondering. The following vignettes come from a study of the relation between proving and knowing of 9th graders (Miyakawa, 2005, p. 225). The two students, L&J, are solving construction and recognition reflective symmetry tasks:

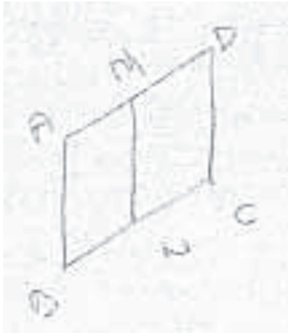
> *Problem:* construction of the symmetrical of a segment:

28. J: it's ok there.
29. L: a right angle…, then, we take the compass like that… you see?
30. J: yes.
31. L: oups, wait… if we fold it like that… yes it fits, it's ok
32. J: hum.



Problem: to recognize a relation of symmetry

Given hypothesis:
$ABCD$ parallelogram
M middle of $[AD]$
N middle of $[BC]$



L&J Proof:
A is the symmetric point of D with respect to M, because A and D are at the same distance to M and the 3 points are aligned.
B is the symmetric point of C with respect to N, because B and C are at the same distance to N and the 3 points are aligned.
Conclusion: AB and DC are symmetrical with respect to the line MN

148. J: yes, that's the same as before. If, if M is the middle of AD, and N is the middle of BC. MN ...
[...]
153. L: that means that somewhere, the right angle and all that, it doesn't exist anymore.
154. J: hum…
155. L: so, so, wait, M is ..., shit, A is the symmetrical of D
156. J: ah, yes, we say the same thing.
157. L: well yes.

The case of L&J evidences the critical role of controls on the decisions and actions. There are both visual controls related to symmetry as paper folding and controls associated to the use of instruments (problem 1), and controls based on geometrical properties and based on a global common sense of symmetry (which obliterates the geometrical control). The issue that is there illustrated is not a lack of logic or the absence of knowledge but that of the restructuration of knowledge. Even a mathematician, in everyday life, first assess perceptively and globally the symmetry of an installation, before using his mathematical competencies.

5. Conception, Explanation and Argumentation

We have personal and daily experiences of using the same piece of mathematical knowledge in different ways depending on the situation and on the context. Without noticing, we could use decimal numbers as a pair of integers when they represent a price to be paid, or as integers equipped with a dot depending on the choice of the unit. Both are not congruent to the mathematical meaning of the corresponding concept. In the case of students, it can lead to errors in certain situations. We used to see there the evidence of "misconceptions". However, these errors more often than not are the result of the extension of procedures and knowledge valid within a certain domain but faulty beyond it.

I proposed to unify the facets of a same piece of knowledge within a model constructed on the notion of "conception" to denote an understanding which has the properties of a piece of knowledge within a certain domain of validity. Once *a set of problems* has been specified as being its domain of validity, a conception can be characterized by three joint and linked sets: a *set of semiotic tools*, a *set of operators* and a *control structure* that allows one to assess, choose and decide (Balacheff, 2013).

Control structures regulate problem-solving processes from its very beginning until the final decision of its successful end. Thus, the validity of a solution is fundamentally dependent on the conceptions. At the early stages, students may rely on a combination of pragmatic and knowledge-based criteria, which is not in line with the mathematical norms. But we know that these norms evolved over history, as they evolve with the learning of mathematics. Then, we may agree on the following claims:

- The validation of a statement, depends on the means of representing, linking and processing the objects at stake, as well as on the associated means of control.
- The rationality of students is built up from the very first activities in the mathematics classroom, which enable them to enter into a validation approach well before the complete formalisation of mathematical objects.

Then, the collective activities in the classroom, regulated by the teacher as a mathematical referent, imposes a *socio-mathematical norm* (Cobb and Yackel, 1996) which may not comply to the canonical ones, but which can be accepted provided it respects minimal conditions (Pedemonte, 2005, p. 17):

- Availability of theorems corresponding to the operators;
- Existence of a mathematical framework that can be substituted for the conception and provide the theoretical basis — i.e. objects and a system of deduction and accepted principles.

6. Proving and Knowing, A Dialectic Interaction

6.1. Empirical and intellectual proofs

The mutual dependence of representation systems and control structures makes it necessary to distinguish different types of proof in order to account for their differences and their evolution. The classification I proposed at the end of the 1980s had this objective. It is often interpreted as a sequence of “stages”, which it is not. The observations, on which it was based, evidenced that students accept a type of proof according to their conceptions *and* according to their perception of the situation. This dependence is particularly obvious when dealing with counterexamples. Different validation approaches can be identified in the course of solving a problem or in the course of a contradictory debate. The stakes of the social interactions or those of the situation may even lead to the obliteration of argumentation in favour of persuasion. Eventually, a type of proof is less an information on the student than on *the student in a situation at a given moment in his/her mathematical history.*

In the early grades, problems preferably deal with familiar or concrete experience. The more the students advance in their schooling the less such a context is available, mathematics becoming more and more abstract. But, having or not having access to a concrete referent is a characteristic of a learning situation that play a central role in setting up the problem of validation. The possibility to execute a decision or to satisfy an assertion give access to pragmatic validations. When this access is not possible, validations are necessarily intellectual. So, the production of intellectual proofs requires, among other things, the linguistic or semiotic expression of objects and their relations.

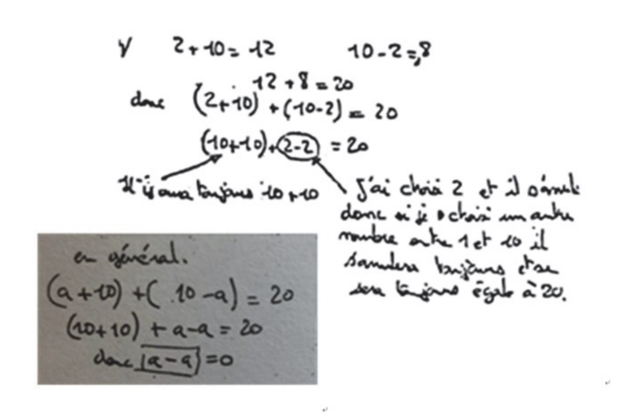
The passage from naive empiricism to mathematical proof can, as it were, describe the movement of the learning of proof in the mathematics classroom. This movement is that from a pragmatic approach to a theoretical one, and thus of an evolution of *the reading of the learning situations* in which the mathematical activity unfolds and the status of the mobilized knowledge evolves.

6.2. The pivotal role of generic examples

The generic example consists in the elicitation of the reasons for the validity of a statement by the realization of operations or transformations on an object present not for itself but as a representative of a class of objects sharing the same characteristics. The formulation puts highlights and structures these characteristics of the class while remaining attached to the exhibition of one of its representatives without depending on its singular properties. This the process by which we see the general in the particular.

The generic example is on the border between pragmatic and intellectual proof, which crossing is brought about by the awareness of the generic character of the case.

Here is a vignette illustrating the generic character of the example used by the student is attested. This come from a replication of the work of Alan Bell which I replicated at the beginning on my research.



"*donc*" translation "so"

- There will always be $10 + 10$
- I have chosen 2 and it nullifies itself, so if I choose another number between 1 and 10, it always nullify itself and always equal.

In the grey box the final version of the proof.

Fig. 2. Choose any number between 1 and 10. Add it to 10 and write down the answer. Take the first number away from 10 and write down the answer. Add your two answers.
1. What result do you get?
2. Try starting with other numbers. Do you get the same result?
3. Will the result be the same for all starting numbers?
4. Explain why your answer is right. (Bell 1976 p.40)

What is written completes the movement towards a representation that gives an account of generality, while at the same time retaining a control over the thread of the writing that reflects that of the construction of the solution; thus, one can understand the strange "therefore $a - a = 0$".

The challenge for the teacher who may use examples in his teaching, is being precise in making the generic character of the case. As a probationary means, a generic example is not just an example.

6.3. The didactical challenge

The early learning of proof in the mathematics classroom requires the creation of a situation in which students are likely to make a problem their own in order to take responsibility for the solution they propose. Research projects have explored various approaches from open inquiry-based learning situation to designing specific situations. They imply a demanding commitment of the teacher to implement them and to maintain a mathematical meaning of the activity while stepping back in order to respect the students' autonomy. The weak point is wrapping up the situation moving from a debate on the validity of a statement, to a debate on the nature and structure of the argumentation itself as an object whose explicit characteristics condition its admissibility as a proof. In other words, the question of the validity of the solution of the problem precisely at stake must be surpassed to leave room for that of the criteria of truth, which is nothing other than laying the foundations of the production of mathematical knowledge.

The validation of a mathematical statement does not get its legitimacy from the compliance to logic and from the sole status of the statements mobilized, but from that of the set of statements to which they are linked within a structured whole: a theory that must be recognized as such.

In effect, the reference to an explicit theoretical framework as a context for mathematical activity is present in many researches but has not been thematized until Alessandra Mariotti's (2001; 1997) proposal to define a "theorem" as the system of mutual relations between three components: a statement, its proof and the theory within which this proof makes sense.

Designing situations that allow to realize these conditions is the main problem we are facing. Among them is taking argumentation, the heart of problem solving, as an object for understanding and learning what a proof is in mathematics.

7. Mathematical Argumentation

7.1. The complexity of the epistemological genesis of mathematical argumentation

There are various forms of validation which weights change along a continuum from the statement of a problem to the communication of its solution according to a norm in force. Their interactions with and their dependencies on the underpinning conceptions a system whose nature determines that of mathematics itself.

During the last two decades, educational decision makers have sought to establish a relationship with mathematics that is closer to the epistemological characteristics of the discipline. Thus, the acquisition of knowledge was completed by that of "competences" among which curricula designate reasoning and mathematical communication. Could the rather broad definition of these prompts the emergence of an activity that gives depth to the mathematical discourse and thus bring to life in the classroom a real little mathematical society? Of course, there is no clear-cut answer.

Proof situations must have the characteristics of situations of validation with the additional constraint of creating an intrinsic need for the analysis, certification and institutionalization of the means of proof in the collective framework of the class. But while we know rather precisely what a proof should be in terms of a learning objective at the end of the compulsory school, there is no shared characterization that can serve as a reference in the course of the schooling that precedes it. Thus, a major theme is the characterisation of *mathematical argumentation* as a legitimate means of establishing truth and as a precursor to the learning of mathematical proof.

A mathematical argumentation must be potentially admissible with respect to the norms of the mathematics classroom, i.e. be accepted as proof by the class and confirmed by the teacher. This is a minimal condition taking into account the social dimension. I propose to start from the Andreas Stylianides definition (Stylianides, 2007, p. 291):

> A proof is a mathematical argument, a connected sequence of statements for or against a mathematical assertion, with the following characteristics:
>
> 1. It uses statements accepted by the class community (set of accepted statements) that are true and available without further justification;
> 2. It uses forms of reasoning (modes of argumentation) that are valid and known to the class community, or within its conceptual reach;
> 3. It is communicated using forms of expression (modes of argument representation) that are appropriate and known to the classroom community, or within its conceptual reach.

For the most part, this proposal is congruent with the common definition of proof. Its interest lies in highlighting three characteristics which correspond to three problems that need to be solved in teaching. The first one is the problem of the creation of a reference, the form of which must be modelled and the conditions of creation specified. The second and third distinguish two aspects of argumentation, its nature (types of argumentation) and its expression (modes of representation of arguments). These two characteristics are in fact intertwined in the process of producing argumentation: reasoning and argumentation are constrained by the means of representation, the language skills, and the level of the conceptions mobilize and shared (e.g. the case of the generic example).

However, although the historical roots of mathematical proof would give it legitimacy, *the concept of mathematical argumentation will be a didactic concept* and not the transposition of a mathematical one, unless we consider that the “social” function of the latter, within the scientific community, is constitutive of it. This would be an epistemological as well as a theoretical error: although being the product of a human activity that is the object of a certification at the end of a social process, a mathematical proof is independent of a particular agent. The normalization of proof in mathematics, besides the institutional character of its theoretical reference, has required its depersonalization, decontextualization and timelessness. On the contrary argumentation is intrinsically carried by an agent and is dependent on the circumstances of its production.

The characteristics of mathematical argumentation must not only allow it to be distinguished from other argumentation practices and norms in order to guarantee the transition towards the norm of mathematical proof, it has also to be effective when it comes to arbitrating the students’ proposals. Moreover, the mathematical argumentation must satisfy the requirements of institutionalization. It is a difficult and delicate problem at the elementary levels, the recognition of its mathematical character cannot be reduced to assessing its form. How, for example, to arbitrate a generic example which puts in balance the general and the specific, whose equilibrium is found at the end of a contradictory debate seeking an agreement as little as possible tainted by compromise?

Proof is both the foundation and the organizer of knowledge. It contributes to reinforcing its evolution and to providing tools for its organization. In teaching, it legitimizes new knowledge and constitutes a system: knowledge and proof linked together make up “theory”. The institutionalization of proof places explicit validation under the arbitration of the teacher who is ultimately the guarantor of its mathematical character. This social dimension, in the sense that scientific functioning depends on a constructed and accepted organization, is at the heart of the difficulty of teaching proof in mathematics.

7.2. When is an argumentation mathematical?

One engages in looking for a proof of a statement if there are reasons, based on her or his conceptions, to support its truth. This condition being verified, the statement deserved its recognition as a conjecture. This observation led me to propose a characterization of conjecture which mirror the characterization of theorem:

$$\text{Conjecture} = \{\text{conception}, \text{statement}, \text{argumentation}\}.$$

Establishing the validity of the conjecture requires reasoning and its formulation including shaping a sentence to express its statement. These constructions evolve along with the problem-solving process up to the point where *an explanation of the truth* is established *in the eyes of the problem-solver*, individual or collective, which could work at least as an argumentation for others, possibly even being accepted as an explanation.

In a proper mathematical activity, the expected future of a conjecture is to be transformed into a theorem. But in early grades the knowledge of reference is not organised into a theory and the structure of the proof does not conform mathematical norms. Moreover, in mathematics teaching not all true statements become theorems: a theorem is in the classroom an institutionalized statement which can be used without producing again its proof. For this reason, at the grade levels considered, I suggest to refer to the validated conjecture as valid statement, and to characterize it by the triplet:

$$\begin{aligned}&\text{Valid statement}\\&\quad = \{\text{knowledge base}, \text{sentence}, \text{mathematical argumentation}\}.\end{aligned}$$

This puts on the fore the role of the knowledge base which is meant to be the same as the role of theory in the case of theorem, that is the reference where it is legitimate to take statements for constructing the argumentation. More often than not, this reference exists but it is left as an implicit clause of the didactical contract; it is the statements which have been stamped as such in previous lessons. This is more a tool-box (Reid, 2011, p. 26) but it could play a role analogous to that of a theory, congruent to the Hans Freudhental (1973, p. 390) idea of a *local organization* which can be regarded mathematical if it is limited enough so its consistency and its domain of validity can be pragmatically ensured. An example of such a reference being explicit to the students could be the quasi-axiomatized[10] geometry of 8th text books in Japan based on a "deliberate choice" of fundamental properties, and their local organization as a system (Miyakawa, 2016). Another example, could come from the use of microworlds which have the specific property to evolve from a few tools and primitives to complex objects with the knowledge of the student (Mariotti, 2001).

Associating different semiotic registers, a *mathematical argumentation* is a multimodal text which does not stand alone: it is built around a sentence and contextualised by a state of knowledge. Its characterization requires that of each of these components:

- *A knowledge base* — explicit, established by and for the classroom community;
- *A sentence* — linguistically appropriate, semantically adequate, of a general stance;
- *An argumentation* — ethically minded, formally coherent, congruent to students' conceptions linking the sentence to the knowledge base.

Generic examples and thought experiments are candidate forms of such argumentations. Becoming a sociomathematical norm, mathematical argumentation shall turn the elementary classroom into a mathematical society, although situated and provisory. It will prepare the K-9 graders to move from the position of practitioners to that of a theoretical approach of mathematics as a science. However, reaching this objective is a challenge for the mathematics education community. One of its aspects was highlighted by Patricio Herbst as we were co-authoring a paper, it deserves the concluding words:

> "Classroom activities are not mathematical performances just because the classroom is a mathematics classroom and not only when their performance is faithful to a mathematically vetted score, yet the observer needs means to support the claim that a classroom activity is a mathematical performance even when they may not have used an accepted definition, a conventional symbol, or a syntactically valid proof." (Herbst and Balacheff, 2009)

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[10] "Quasi" means that certain properties are introduced by observation or accepted.

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